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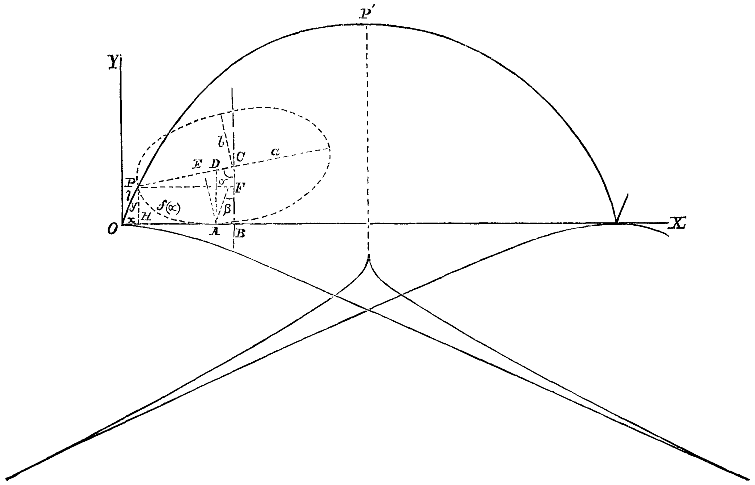
Note on a Roulette.

By A. V. Lane, Ph. D.



Let us imagine an ellipse of semi-axes a and b to roll on a right line, taking one extremity of its major axis as the generating point.

With the notation of the diagram, we have

$$x = OH = OA + AB - PF = \text{arc } PA + r\sin\beta - a\sin\alpha. \tag{1}$$

$$y = HP = CB - CF = r\cos\beta - a\cos\alpha. \tag{2}$$

Now the sub-normal in the ellipse equals $\frac{b^2}{a^2}$ multiplied by the abscissa of the point of contact. $\therefore\ DE = \frac{b^2}{a^2}CE$. But $\frac{EA}{CE} = \tan(\alpha - \beta)$ and $\frac{EA}{DE} = \tan\alpha$. $\therefore\ \frac{\tan(\alpha - \beta)}{\tan\alpha} = \frac{b^2}{a^2}$. Whence

$$\sin\beta = \frac{(a^2 - b^2)\sin\alpha\cos\alpha}{(a^4\cos^2\alpha + b^4\sin^2\alpha)^{\frac{1}{2}}} \tag{3}$$

$$\sin(\alpha - \beta) = \frac{b^2\sin\alpha}{(a^4\cos^2\alpha + b^4\sin^2\alpha)^{\frac{1}{2}}}. \tag{4}$$

From the equation of the ellipse, we find

$$r = \frac{ab}{[a^2 \sin^2(\alpha - \beta) + b^2 \cos^2(\alpha - \beta)]^{\frac{1}{2}}} = \frac{(a^4 \cos^2\alpha + b^4 \sin^2\alpha)^{\frac{1}{2}}}{(a^2 \cos^2\alpha + b^2 \sin^2\alpha)^{\frac{1}{2}}}. \tag{5}$$

$$\text{The arc } PA = \int_0^\alpha \frac{a^2 b^2 d\alpha}{(a^2 \cos^2\alpha + b^2 \sin^2\alpha)^{\frac{3}{2}}} = f(\alpha). \tag{6}$$

Making the necessary substitutions from (3)–(6) in (1) and (2) and letting $c^2 = a^2 - b^2$, we have, as the equation of the roulette,

$$\begin{cases} x = f(\alpha) + \dfrac{c^2 \sin\alpha \cos\alpha}{(a^2 \cos^2\alpha + b^2 \sin^2\alpha)^{\frac{1}{2}}} - a \sin\alpha & (7) \\ y = (a^2 \cos^2\alpha + b^2 \sin^2\alpha)^{\frac{1}{2}} - a \cos\alpha. & (8) \end{cases}$$

For brevity let ( ) represent $a^2 \cos^2\alpha + b^2 \sin^2\alpha$. Then

$$\frac{dx}{d\alpha} = (\ )^{\frac{1}{2}} - a \cos\alpha = y, \tag{9}$$

$$\frac{dy}{d\alpha} = -\frac{c^2 \sin'\alpha \cos\alpha}{(\ )^{\frac{1}{2}}} + a \sin\alpha = f(\alpha) - x, \tag{10}$$

$$\frac{dy}{dx} = \frac{a + \cos\alpha (\ )^{\frac{1}{2}}}{\sin\alpha (\ )^{\frac{1}{2}}} = \frac{f(\alpha) - x}{y}, \tag{11}$$

$$\frac{d^2y}{dx^2} = -\frac{(\ )^{\frac{3}{2}} - a \cos\alpha [a^2 - 2(\ )]}{\sin^2\alpha (\ )^{\frac{3}{2}} [(\ )^{\frac{1}{2}} - a \cos\alpha]}. \tag{12}$$

$$\text{The radius of curvature } \rho = \pm \frac{[a^2 + (\ ) + 2a \cos\alpha (\ )^{\frac{1}{2}}]^{\frac{3}{2}} [(\ )^{\frac{1}{2}} - a \cos\alpha]}{\sin\alpha \{(\ )^{\frac{3}{2}} - a \cos\alpha [a^2 - 2(\ )]\}}. \tag{13}$$

$$\rho_0 = \rho_{360} = 0 . \rho_{180} = \frac{4a^3}{2a^2 - b^2} . \rho_{180} - 2a = \frac{2ab^2}{2a^2 - b^2}.$$

With the same axes and origin, but calling x' and y' the coordinates of the evolute, we obtain its equation by substituting from (7), (8), (11) and (12) in the expressions

$$x' = x - \frac{1 + \left(\dfrac{dy}{dx}\right)^2}{\dfrac{d^2y}{dx^2}} \cdot \frac{dy}{dx}, \tag{14}$$

$$y' = y + \frac{1 + \left(\dfrac{dy}{dx}\right)^2}{\dfrac{d^2y}{dx^2}}. \tag{15}$$

Whence the equation of the evolute is

$$\left\{\begin{aligned} x' &= f(\alpha) + \frac{a^2b^2 \sin\alpha\,[a + \cos\alpha\,(\;)^{\frac{1}{2}}]}{(\;)^{\frac{1}{2}}\{(\;)^{\frac{3}{2}} - a\cos\alpha\,[a^2 - 2(\;)]\}}, && (16)\\ y' &= -\frac{a^2b^2\sin^2\alpha}{(\;)^{\frac{3}{2}} - a\cos\alpha\,[a^2 - 2(\;)]} && (17)\end{aligned}\right.$$

$$y'_0 = y'_{360} = 0\,.\, y'_{90} = y'_{270} = -\frac{a^2}{b}\cdot y'_{180} = -\frac{2ab^2}{2a^2 - b^2}\,.$$

Let $[\;]$ represent $(\;)^{\frac{3}{2}} - a\cos\alpha\,[a^2 - 2(\;)]$.

Let $\}\;\{$ represent $\cos\alpha\,(\;)^{\frac{1}{2}}\,[2a^2 + c^2\sin^2\alpha] - a^3 + 3a^3\cos^2\alpha + 2a\sin^2\alpha\,(\;)$.

Then $$\frac{dx'}{d\alpha} = a^2b^2\,\frac{[a + \cos\alpha\,(\;)^{\frac{1}{2}}]}{(\;)^{\frac{1}{2}}}\cdot\frac{\{\;\}}{[\;]^2}, \tag{18}$$

$$\frac{dy'}{d\alpha} = -a^2b^2\sin\frac{\{\;\}}{[\;]^2}, \tag{19}$$

$$\frac{d^2y'}{d\alpha^2} = -a^2b^2\,\frac{\cos\alpha\,[\;]\,\{\;\} + \sin\alpha\,[\;]\,\dfrac{d\,\{\;\}}{d\alpha} - 2\sin\alpha\,\{\;\}\,\dfrac{d[\;]}{d\alpha}}{[\;]^3}, \tag{20}$$

$$\frac{dy'}{dx'} = -\frac{1}{\dfrac{dy}{dx}} = -\frac{\sin\alpha\,(\;)^{\frac{1}{2}}}{a + \cos\alpha\,(\;)^{\frac{1}{2}}}, \tag{21}$$

$$\frac{d^2y'}{dx'^2} = -\frac{[\;]^3}{a^2b^2\,[a + \cos\alpha\,(\;)^{\frac{1}{2}}]^3\{\;\}}\,. \tag{22}$$

To determine maximum and minimum values of y', we put $(19) = 0$. Whence $\sin\alpha = 0$, $\cos\alpha = \pm 1$; also $\{\;\} = 0$. This last gives (letting $w = \cos^2\alpha$) a biquadratic in w, two of whose roots are unity. Dividing by $(w - 1)^2$, we have the quadratic

$$w^2 - \frac{8a^4 - 3a^2b^2 - b^4}{c^2(3a^2 + b^2)}\,w + \frac{a\; - 4a^4b^2 + 4a^2b^4}{c^4(3a^2 + b^2)} = 0,$$

whose roots are

$$\frac{1}{2c^2(3a^2 + b^2)}\,\{8a^4 - 3a^2b^2 - b^4 \pm \sqrt{52a^8 - 4a^6b^2 - 39a^4b^4 - 10a^2b^6 + b^8}\} = \frac{1}{P}\,\{M \pm \sqrt{N}\}.$$

$\frac{M}{P} > 1$, and the plus sign cannot therefore be used before the radical as it would make $\cos\alpha > 1$. $M > 0$, $M - \sqrt{N}$ is always real, equals 0 when $\frac{a}{b} = \sqrt{2}$, and, for all other values of this ratio, is positive. Let $\frac{a}{b} = k$ and $+\frac{1}{2}\sqrt{1 + \sqrt{13}} = +1.073027 + = \mu$. Then $w < 1$ for values of $k > \mu$, equals 1 when k equals μ and equals 0 when $k = \sqrt{2}$; giving in these cases possible values to $\cos\alpha$. But when $k < \mu$, $w > 1$ and $\cos\alpha$ impossible. So the values of y' to be investigated are those corresponding to

$$\cos\alpha = \pm 1 \text{ (from } \sin\alpha = 0),$$

$\cos\alpha = -1$ (derived from $(w-1)^2 = 0$ or from $w = \frac{1}{P}\{M - \sqrt{N}\}$ when $k = \mu$; $\cos\alpha = +1$ will not reduce $\{\ \}$ to 0),

$$\cos\alpha = -\sqrt{\frac{M-\sqrt{N}}{P}} \text{ when } k > \mu \text{ and } < \sqrt{2},$$

$$\cos\alpha = 0 \text{ when } k = \sqrt{2},$$

$$\cos\alpha = +\sqrt{\frac{M-\sqrt{N}}{P}} \text{ when } k > \sqrt{2},$$

approaching the limit $+\sqrt{\frac{4-\sqrt{13}}{3}} = +0.362606$—as k approaches ∞.

When $\cos\alpha = +1$, $\frac{d^2y'}{d\alpha^2} = -\frac{b^2}{a}$ and $y' = 0$ is a maximum; or, all other values of y' being negative, this is *numerically* a minimum.

When $\cos\alpha = -1$, $\frac{d^2y'}{d\alpha^2} = -\frac{b}{k}\,\frac{4k^4-2k^2-3}{(2k^2-1)^2}$ and $y' = -\frac{2ab^2}{2a^2-b^2}$ is numerically a maximum when $k < \mu$ and a minimum when $k > \mu$. When $k = \mu$, $\frac{d^2y'}{d\alpha^2} = 0$. But an investigation of the sign of $\frac{d^2y'}{dx'^2}$ shows it to be always negative and therefore the evolute always convex to the axis of X, for values of $k =$ or $< \mu$. This, combined with the fact that (21) shows the tangent to be perpendicular to the axis of X when $\cos\alpha = -1$, whatever the value of k, proves y' numerically a maximum when $\cos\alpha = -1$, $k = \mu$.

Omitting from (20) terms containing $\{\ \}$ as a factor, we have

$$\left[\frac{d^2y'}{d\alpha^2}\right]_{\{\ \}=0} = a^2b^2\sin\alpha^2.\ \frac{\cos\alpha(\ )^{\frac{1}{2}}[2a^3+8ac^2\sin^2\alpha]+b^2(3a^2-b^2)+(5c^2+a^2)c^2\cos^2\alpha-4c^4\cos^4\alpha}{(\ )^{\frac{1}{2}}[\ ]^2}.$$

This is clearly positive for positive values of $\cos\alpha$ and for $\cos\alpha = 0$, $k = \sqrt{2}$. It is also positive when $\cos\alpha$ is negative, k lying between $\sqrt{2}$ and μ.

Hence, for all values of $\cos\alpha$ derived from $\{\ \} = 0$, y' is numerically a maximum ($\cos\alpha < 0$, $k = \mu$ has already been shown to give such a value to y'). This maximum disappears when $k =$ or $< \mu$, because the value derived from $\{\ \} = 0$ for $\cos\alpha$ is then > 1.

The length of the half branch of the evolute may be found by substituting in (13) that value of α which makes y' a numerical maximum, doubling and then subtracting ρ_{180}.

The limiting forms of the roulette are the cycloid ($k = 1$) and the semicircle* ($k = \infty$); of the evolute, a cycloid (equal to the involute) and a point (the centre of the involute).

To rectify the roulette—

Substituting in the formula $dl = dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}$, we find

$$dl = \left[\frac{b^2(a^2+b^2)-(b^4+a^2b^2-2a^4)\cos^2\alpha}{(\ )} - 2a^3\cos\alpha\,(\ )^{-\frac{1}{2}}\right]^{\frac{1}{2}} d\alpha.$$

Let $\cos\alpha = u$, $d\alpha = -\frac{du}{\sqrt{1-u^2}}$, $f = b^2(a^2+b^2)$, $g = b^4 + a^2b^2 - 2a^4$, $h = a^2 - 2b^2$; then

$$dl = -\left[\frac{f-gu^2}{b^2+hu^2-c^2u^4} - 2a^3.\frac{u}{1-u^2}(b^2+c^2u^2)^{-\frac{1}{2}}\right]^{\frac{1}{2}} du.$$

Now $$\frac{f-gu^2}{b^2+hu^2-c^2u^4} = A + Cu^2 + Eu^4 + \text{etc.}$$

where $$A = \frac{f}{b^2},\ C = -\frac{1}{b^2}\left(g + \frac{fh}{b^2}\right),$$

$$E = \frac{1}{b^4}\left(c^2f + gh + \frac{fh^2}{b^2}\right),\ \text{etc.}$$

The n^{th} coefficient of this series may be formed thus—first considering the minus signs as plus, add together $\frac{b^2c^2}{b^4}$ times the $(n-2)^{\text{th}}$ coefficient and $\frac{h}{b^2}$ times the $(n-1)^{\text{th}}$ coefficient and multiply this sum by $(-1)^{n-1}$. Again, $(b^2+c^2u^2)^{-\frac{1}{2}} = b^{-1} - \frac{1}{2}b^{-3}c^2u^2 + \frac{3}{8}b^{-5}c^4u^4 -$ etc., and $-2a^3.\frac{u}{1-u^2}(b^2+c^2u^2)^{-\frac{1}{2}} = Bu + Du^3 + Fu^5 +$ etc., where $B = -2a^3b^{-1}$, $D = -2a^3(-\frac{1}{2}b^{-3}c^2 + b^{-1})$, $F = -2a^3(\frac{3}{8}b^{-5}c^4 - \frac{1}{2}b^{-3}c^2 + b^{-1})$, etc.

The n^{th} coefficient of this series may be formed by multiplying $-2a^3$ by the sum of n coefficients of the preceding series.

We may now write $$dl = -(A+R)^{\frac{1}{2}}du,$$

where $$R = Bu + Cu^2 + Du^3 + \text{etc.}$$

Expanding and substituting for R its value, we have

$$dl = A'du + B'udu + C'u^2du + D'u^3du + E'u^4du + \text{etc.}$$

where $$A' = -A^{\frac{1}{2}},\ B' = -\tfrac{1}{2}A^{-\frac{1}{2}}B,\ C' = -\tfrac{1}{2}A^{-\frac{1}{2}}C + \tfrac{1}{8}A^{-\frac{3}{2}}B^2,$$

$$D' = -\tfrac{1}{2}A^{-\frac{1}{2}}D + \tfrac{1}{8}A^{-\frac{3}{2}}(2BC) - \tfrac{1}{16}A^{-\frac{5}{2}}B^3,$$

$$E' = -\tfrac{1}{2}A^{-\frac{1}{2}}E + \tfrac{1}{8}A^{-\frac{3}{2}}(2BD + C^2) - \tfrac{1}{16}A^{-\frac{5}{2}}(3B^2C) + \tfrac{5}{128}A^{-\frac{7}{2}}B^4,\ \text{etc.}$$

Integrating and determining the constant from $\alpha = 0$, $u = +1$, $l = 0$, we have

$$l = A'u + \frac{B'u^2}{2} + \frac{C'u^3}{3} + \frac{D'u^4}{4} + \frac{E'u^5}{5} + \text{etc.} - A' - \frac{B'}{2} - \frac{C'}{3} - \frac{D'}{4} - \frac{E'}{5} - \text{etc.}$$

For $\alpha = \Pi$, $u = -1$, $l = \text{arc } OP' = \frac{1}{2}$ branch. Then arc $OP' = -2\left[A' + \frac{C'}{3} + \frac{E'}{5} + \text{etc.}\right]$.

Calling S the area included between a portion of the curve, beginning at the origin of coordinates, the X axis and the ordinate, we have

$$S = \int y dx = \int y^2 d\alpha$$
$$= 2a^2 \int \cos^2 \alpha d\alpha + b^2 \int \sin^2 \alpha d\alpha - 2a \int (\ )^{\frac{1}{2}} \cos \alpha d\alpha .$$

The first integral equals $\frac{1}{2} \alpha + \frac{1}{4} \sin 2\alpha$, the second equals $\frac{1}{2} \alpha - \frac{1}{4} \sin 2\alpha$. To integrate the last term—expanding we have

$$(\ )^{\frac{1}{2}} \cos \alpha d\alpha = b \cos \alpha d\alpha + \frac{c^2 \cos^3 \alpha d\alpha}{2b} - \frac{c^4 \cos^5 \alpha d\alpha}{8b^3} + \text{etc.}$$
$$= b \cos \alpha d\alpha + \frac{c^2}{2^3 b} (\cos 3\alpha d\alpha + 3 \cos \alpha d\alpha)$$
$$- \frac{c^4}{2^7 b^3} (\cos 5\alpha d\alpha + 5 \cos 3\alpha d\alpha + 10 \cos \alpha d\alpha) + \text{etc.}$$

Integrating this and substituting, we have $S = \frac{1}{2} (2a^2 + b^2) \alpha + \frac{1}{4} (2a^2 - b^2) \sin 2\alpha$

$$- 2a \left[b \sin \alpha + \frac{c^2}{2^3 b} (\tfrac{1}{3} \sin 3\alpha + 3 \sin \alpha) - \frac{c^4}{2^7 b^3} (\tfrac{1}{5} \sin 5\alpha + \tfrac{5}{3} \sin 3\alpha + 10 \sin \alpha) + \text{etc.} \right]$$

Making $\alpha = 2\Pi$, we have the area for one branch equal to $\Pi (2a^2 + b^2)$.

Calling V the volume generated by the revolution of S about the X-axis, we have

$$V = \Pi \int y^2 dx = \Pi \int y^3 d\alpha$$
$$= \Pi \int [(\ )^{\frac{3}{2}} d\alpha - 3ab^2 \cos \alpha d\alpha - a (3c^2 + a^2) \cos^3 \alpha d\alpha + 3a^2 (\ )^{\frac{1}{2}} \cos^2 \alpha d\alpha] .$$

Expanding the irrational expressions and transforming the higher powers of $\cos \alpha$ to cosines of multiples of α as before, we find

$$\frac{V}{\Pi} = \frac{b}{4} (3c^2 + 4b^2) \alpha + \frac{3bc^2 \sin 2\alpha}{2^3} + \frac{3c^4}{2^6 b} (\tfrac{1}{4} \sin 4\alpha + \tfrac{4}{2} \sin 2\alpha + 3\alpha) - \text{etc.}$$
$$- 3ab^2 \sin \alpha - \frac{3ac^2 + a^3}{2^2} (\tfrac{1}{3} \sin 3\alpha + 3 \sin \alpha)$$
$$+ 3a^2 \Big[\tfrac{1}{2} b\alpha + \tfrac{1}{4} b \sin 2\alpha + \frac{c^2}{2^4 b} (\tfrac{1}{4} \sin 4\alpha + \tfrac{4}{2} \sin 2\alpha + 3\alpha)$$
$$- \frac{c^4}{2^8 b^3} (\tfrac{1}{6} \sin 6\alpha + \tfrac{6}{4} \sin 4\alpha + \tfrac{15}{2} \sin 2\alpha + 10\alpha) + \text{etc.} \Big] .$$

For $\alpha = 2\Pi$, we have the volume thus generated by the area of one branch equal to

$$\Pi^2 \left[\tfrac{1}{2} b (3a^2 + b^2) + \frac{18c^4}{2^6 b} - \text{etc.} + 3a^2 \left(b + \frac{bc^2}{2^4 b} - \frac{20c^4}{2^8 b^3} + \text{etc.} \right) \right] .$$

UNIVERSITY OF TEXAS, AUSTIN, July, 1885.